National Qualifications 2018

X702/77/11

Mathematics of Mechanics

TUESDAY, 29 MAY

1:00 PM – 4:00 PM

Total marks — 100

Attempt ALL questions.

You may use a calculator.

Full credit will be given only to solutions which contain appropriate working.

State the units for your answer where appropriate. Any rounded answer should be accurate to three significant figures (or one decimal place for angles in degrees) unless otherwise stated.

Write your answers clearly in the answer booklet provided. In the answer booklet you must clearly identify the question number you are attempting.

Use **blue** or **black** ink.

Before leaving the examination room you must give your answer booklet to the Invigilator; if you do not, you may lose all the marks for this paper.

FORMULAE LIST

Newton's inverse square law of gravitation

$$F = \frac{GMm}{r^2}$$

Simple harmonic motion

$$v^2 = \omega^2(a^2 - x^2)$$

$$x = a\sin(\omega t + \alpha)$$

Centre of mass

Triangle: $\dfrac{2}{3}$ along median from vertex.

Semicircle: $\dfrac{4r}{3\pi}$ along the axis of symmetry from the diameter.

Standard derivatives	
$f(x)$	$f'(x)$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\ln x$	$\dfrac{1}{x}$
e^x	e^x

Standard integrals	
$f(x)$	$\int f(x)\,dx$
$\sec^2(ax)$	$\dfrac{1}{a}\tan(ax) + c$
$\dfrac{1}{x}$	$\ln|x| + c$
e^{ax}	$\dfrac{1}{a}e^{ax} + c$

Total marks — 100

Attempt ALL questions

Candidates should observe that $g\,\mathrm{m\,s^{-2}}$ denotes the magnitude of the acceleration due to gravity. Where appropriate, take its magnitude to be $9{\cdot}8\,\mathrm{m\,s^{-2}}$.

1. A boat accelerates steadily from a speed of $10\,\mathrm{m\,s^{-1}}$ to $14\,\mathrm{m\,s^{-1}}$ over a distance of $1{\cdot}2$ kilometres.

   The boat continues to accelerate at the same rate for a further two minutes.

   The engines are then put into reverse to produce an immediate deceleration of the same magnitude as the previous acceleration. This brings the boat to rest.

   Calculate the total distance travelled by the boat since it started to accelerate. **5**

2. (a) Find partial fractions for
   $$\frac{13+6x+5x^2}{(1+x)(2-x)(3+x)}.$$ **4**

   (b) Show that $\displaystyle\int_0^1 \frac{13+6x+5x^2}{(1+x)(2-x)(3+x)}\,dx = \ln\frac{a}{b}$ where a and b are positive integers. **3**

3. An object of mass $10\,\mathrm{kg}$ is projected along a rough horizontal surface with an initial speed of $12\,\mathrm{m\,s^{-1}}$. The coefficient of friction between the object and the surface is $0{\cdot}25$.

   After travelling a distance of 20 metres along this rough surface it collides and coalesces with a stationary object of mass $5\,\mathrm{kg}$.

   Find the speed of the combined objects immediately after the collision. **5**

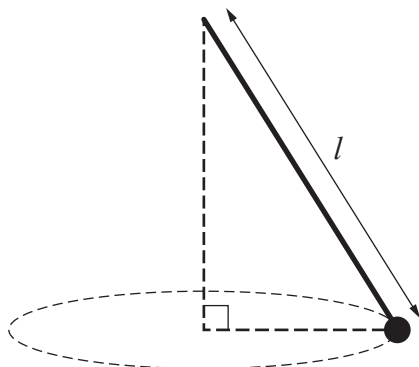4. A function is defined as $f(x)=e^{\sec^2 x}$ where $0 \le x < \dfrac{\pi}{2}$.

   Find the exact value of $f'\!\left(\dfrac{\pi}{4}\right)$. **3**

[Turn over

5. A body of mass m kg is attached to one end of a light inextensible string of length l metres.

The other end of the string is fixed and the body is spun in a horizontal circle so that the path of the string forms a conical pendulum, as shown in the diagram.
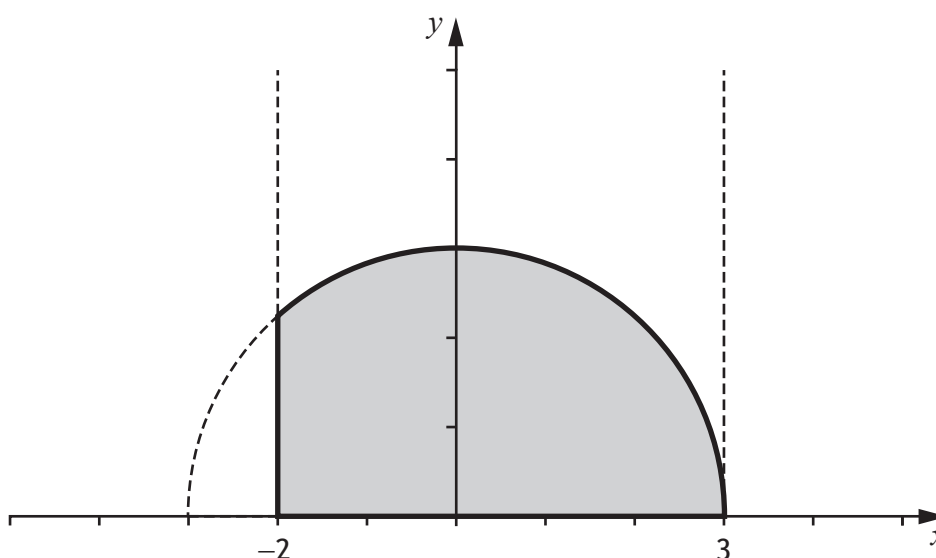

The angular speed of the body is ω radians per second.

Given that the length of the string is double the radius of the horizontal circle, show that

$$\omega^2 = \frac{2g}{\sqrt{3}\,l}.$$

5

6. A part of the graph of $x^2 + y^2 = 9$ is shown in the diagram below.

The shaded area is bounded by the graph, the x-axis and the lines $x = -2$ and $x = 3$. This area is rotated 360° about the x-axis.

Calculate the volume of the solid formed by this rotation.

4

7. A particle is projected from a point A at time $t = 0$ and performs simple harmonic motion with A as the centre of oscillation.

    The amplitude of the motion is 6 metres and period is 10 seconds.

    (a) Calculate the first two times when the particle will be 4 metres from A.   **4**

    (b) Calculate the speed of the particle at the second of these times and comment on its direction.   **2**

8. The motion of a particle is defined by the equations

    $$x = t(t+4) \text{ and } y = t(1-t)^3$$

    where t is the time elapsed since the start of motion.

    Find the speed of the particle when $t = 3$.   **4**

9. A projectile is launched with speed $v\,\mathrm{m\,s^{-1}}$, at an angle θ to the horizontal.

    (a) Show that the horizontal range R of the projectile is given in metres by

    $$R = \frac{v^2 \sin 2\theta}{g}.$$ **4**

    (b) A tennis training device fires balls at the same speed each time, but the angle of projection can vary.

   A ball is fired at 30° to the horizontal and has a range of R metres.

    Another ball is fired at 35° to the horizontal and has a range of $(R+5)$ metres.

    (i) Calculate the initial speed of the balls.   **3**

    (ii) On a particular day, the tennis balls are assisted by a horizontal tailwind of $7\,\mathrm{m\,s^{-1}}$. Find the new range of a ball fired at 35° to the horizontal.   **3**

[Turn over

10. A uniform circular lamina with diameter AB $= 8\,$cm and centre C has mass per unit area m.

Two holes have been made in the lamina as shown in the diagram.
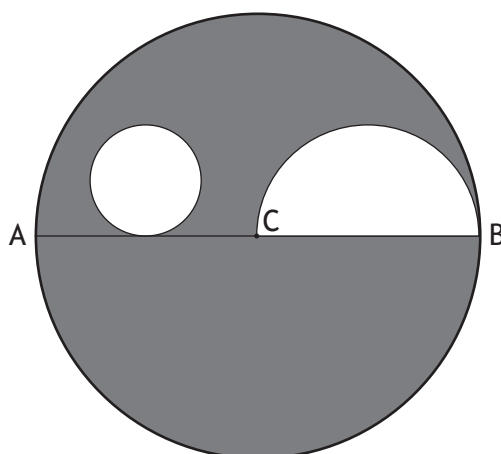
The circular hole has radius 1 cm and touches the line AB at a point 2 cm from A. The other hole is a semi-circle with diameter BC.

(a) Find the position of the centre of mass of the lamina relative to the point A. **6**

(b) The lamina is freely suspended from the point A. Calculate the angle that the line AB makes with the downward vertical. **1**

11. At 08:00 a port official records Boat A at point $(1\cdot2, 1\cdot6)$ and Boat B at $(34\cdot8, 1)$, where the distances are in kilometres relative to the port as an origin.

At 08:06 the official records their points as $(6, 3)$ and $(34, 2\cdot5)$ respectively.

(a) Show that their average velocities over this period can be expressed, in $km\,h^{-1}$, as

$$\mathbf{v}_A = 48\mathbf{i} + 14\mathbf{j} \ \text{ and } \ \mathbf{v}_B = -8\mathbf{i} + 15\mathbf{j}.$$ **2**

(b) (i) Assuming that each boat maintains a constant velocity, show that they are on a collision course. **3**

(ii) Find the location of the collision. **1**

12. A zip wire can be modelled by a taut inextensible cable with a seat attached to it by a light metal rod of length 1·8 metres.

The cable is at an angle of 20° to the horizontal, as shown in the diagram.
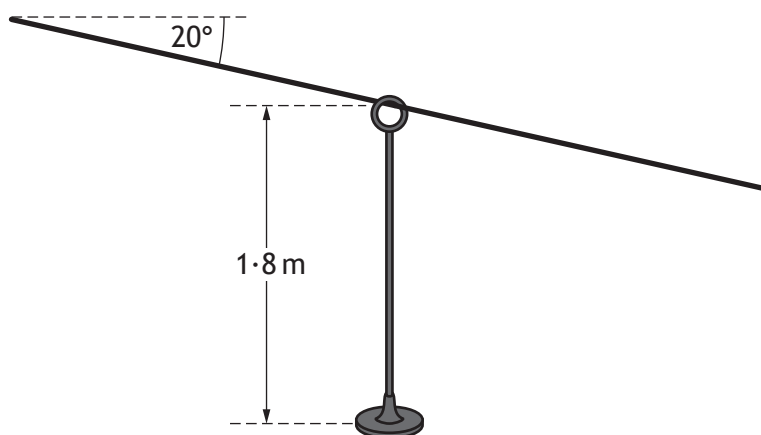
A child sits on a seat at the higher end of the zip wire and is given an initial speed of $2 \, m \, s^{-1}$, parallel to the cable. The combined mass of the child and seat is m kg.

The coefficient of friction between the rod and the cable is 0·3 and the cable is 20 m long.

You may assume that the rod remains vertical as it travels along the cable.

(a) Calculate the speed of the child at the lower end of the zip wire. **4**

At the lower end of the zip wire the top of the rod hits a buffer and stops. The rod then swings through an angle θ from a vertical position before coming momentarily to rest and swinging back.

(b) Calculate the value of θ. **4**

13. Find the exact value of the integral $\displaystyle\int_{0}^{\sqrt{5}} \frac{2x^3}{\sqrt{x^2+4}}\,dx$ using the substitution $u = x^2 + 4$. **6**

14. A bungee jumper of mass 70 kg stands on a bridge 40 metres above a river. The natural length of the bungee cord is 10 metres and it has a modulus of elasticity of 1000 newtons.

If the bungee jumper falls vertically from rest, calculate their height above the water when the cord is fully extended. **5**

[Turn over

15. A spring is attached to a fixed point P. The other end is attached to a block of wood on a smooth horizontal surface as shown in the diagram.
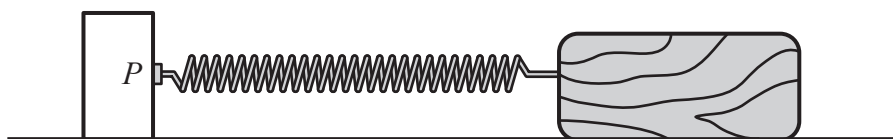
The spring is stretched so that the block of wood moves 1·5 metres from its rest position. The block is then projected with a speed of $0·5\,\text{m s}^{-1}$ towards P at time $t = 0$.

The subsequent motion can be modelled by the differential equation

$$\frac{d^2x}{dt^2} + 0·4\frac{dx}{dt} + 0·04x = 0$$

where x metres represents the displacement from the rest position, and t is measured in seconds.

(a) Solve this second order differential equation and use the initial conditions given to determine an expression for x in terms of t. **5**

(b) Hence calculate how far the block of wood has moved after 2 seconds. **1**

16. Two runners are taking part in a relay race and preparing to hand over the baton. They are running in the same straight line when they exchange the baton.

Runner *P* is running at a constant speed of $12\,\text{m s}^{-1}$ when he decelerates at $4\,\text{m s}^{-2}$ in preparation for the baton change, at which point he must be travelling at $9\,\text{m s}^{-1}$. He continues to decelerate at the same rate until he comes to rest.

Runner *Q* takes the baton 3 seconds after starting running with a constant acceleration. He has achieved a speed of $9\,\text{m s}^{-1}$ when the baton is exchanged and continues to accelerate to a maximum speed of $12\,\text{m s}^{-1}$.

(a) (i) Sketch a velocity/time graph to represent both runners, annotating all relevant points on your graph. **2**

 (ii) For how many seconds has *P* decelerated before baton change? **1**

(b) At the point when the baton is exchanged, *Q* is 0·8 metres ahead of *P*.

How far is *P* behind *Q* when *Q* starts to run? **5**

17. A box of mass m kg is set in motion with an initial impulse I. As it moves along the surface it experiences a resistive force proportional to the square of its velocity $v\,\text{m s}^{-1}$.

By setting up a differential equation, show that the velocity of the box after t seconds can be expressed as $v = \dfrac{mI}{Ikt + m^2}$, where k is a constant and t is measured from the moment of impulse. **5**

[END OF QUESTION PAPER]